

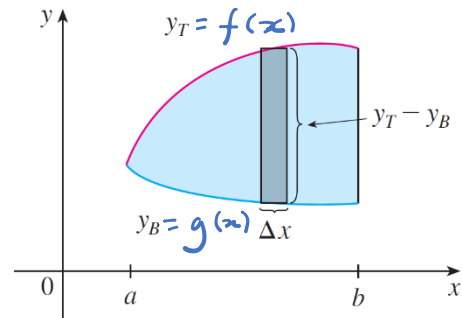
Section 6.1 – Areas Between Curves

- Areas between curves

Region Type I (Vertical): $y_T \geq y_B \quad \forall x \in [a, b]$

$$A = \int_a^b [y_T - y_B] dx$$

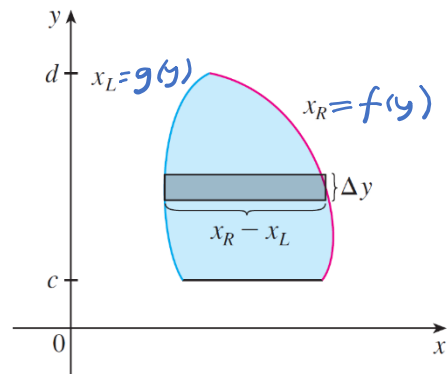
$f(x)$ $g(x)$



Region Type II (Horizontal): $x_R \geq x_L \quad \forall y \in [c, d]$

$$A = \int_c^d [x_R - x_L] dy$$

$f(y)$ $g(y)$



Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution

$$\begin{aligned}
 & \int_0^1 y_T - y_B dx \\
 &= \int_0^1 e^x - x dx = e^x \Big|_0^1 - \frac{x^2}{2} \Big|_0^1 \\
 &= (e - 1) - \frac{1}{2} \\
 &= e - \frac{3}{2}
 \end{aligned}$$

Example 2

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

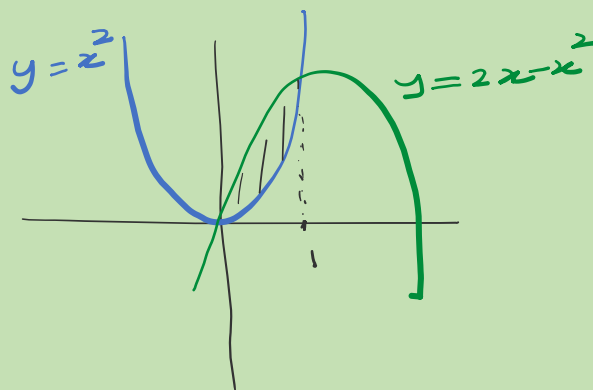
Solution

$$2x - x^2 = x^2$$

$$2x - 2x^2 = 0$$

$$2x(x - 1) = 0$$

$$x = 0 \quad x = 1$$



$$A = \int_0^1 (2x - x^2 - x^2) dx$$

$$= \left. \frac{2x^2}{2} - \frac{2x^3}{3} \right|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

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طرفة y_1 و y_2

$$y_1\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y_2\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Example 3

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

$$x = y + 1$$

$$x = \frac{1}{2}y^2 - 3$$

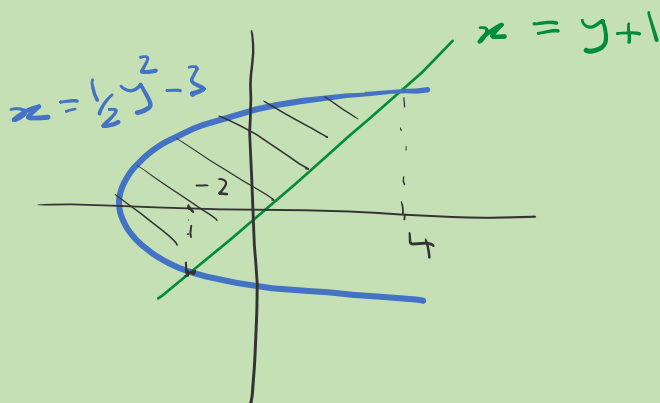
$$y + 1 = \frac{1}{2}y^2 - 3 \Rightarrow \frac{1}{2}y^2 - y - 4 = 0 \quad \times 2$$

$$y^2 - 2y - 8 = 0$$

$$(y - 4)(y + 2) = 0$$

$$y = 4$$

$$y = -2$$



$$x_1(0) = 0 + 1 = 1$$

$$x_2(0) = \frac{1}{2}(0) - 3 = -3$$

$$A = \int_{-2}^4 (y + 1 - (\frac{1}{2}y^2 - 3)) dy$$

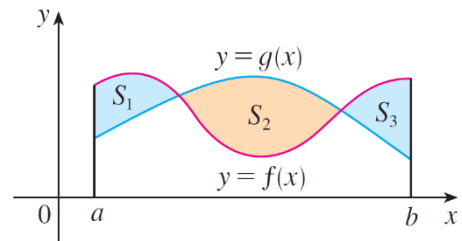
$$= \int_{-2}^4 -\frac{1}{2}y^2 + y + 4 dy = -\frac{1}{2} \frac{y^3}{3} + \frac{y^2}{2} + 4y \Big|_{-2}^4$$

$$= -\frac{4^3}{6} + \frac{4^2}{2} + 4 \cdot 4 - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4 \cdot (-2) \right) = 18$$

- Varying inequality

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx$$



$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) \\ g(x) - f(x) \end{cases}$$

when $f(x) \geq g(x)$

when $g(x) \geq f(x)$

Example 4

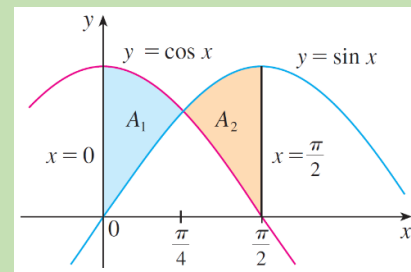
Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution

$$\sin x = \cos x \quad x \in [0, \pi/2]$$

$$x = \frac{\pi}{4}$$

$$A = A_1 + A_2$$



$$= \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} + \left[-\cos x - \sin x \right]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - \left(\sin 0 + \cos 0 \right) \right]$$

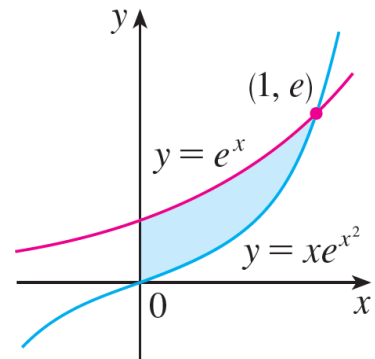
$$+ \left[\left(-\cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right) \right]$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right] + \left[-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] = 2\sqrt{2} - 2$$

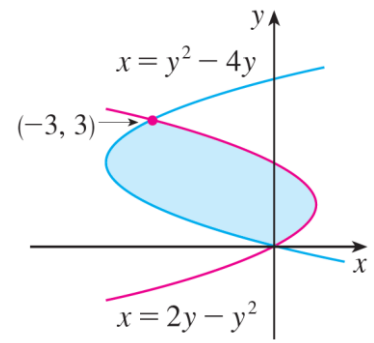
Problems

- Find the area of the shaded region.

(a)



(b)



- Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y . Then find the area of the region.

(a) $y = x^2 - 2x$, $y = x + 4$

(b) $y = 1/x$, $y = 1/x^2$, $x = 2$

- Sketch the region enclosed by the given curves and find its area.

(a) $y = \cos x$, $y = 2 - \cos x$, $0 \leq x \leq 2\pi$

(b) $x = 2y^2$, $x = 4 + y^2$

(c) $y = \sqrt{x-1}$, $x - y = 1$